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Efficient and perfect state transfer in quantum chains

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Abstract

We present a communication protocol for chains of permanently coupled qubits which achieves perfect quantum state transfer and which is efficient with respect to the number of chains employed in the scheme. The system consists of M uncoupled identical quantum chains. Local control (gates, measurements) is only allowed at the sending/receiving end of the chains. Under a quite general hypothesis on the interaction Hamiltonian of the qubits, a theorem is proved which shows that the receiver is able to asymptotically recover the messages by repetitive monitoring of his qubits.

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1. Introduction

Permanently coupled quantum chains have recently been proposed as prototypes of reliable quantum communication lines [1, 2]. The main drawback of these schemes is related with the fact that even in the absence of external noise the fidelity of the transmission is in general not optimal [1, 3–7]. This is due to the dispersion which affects the propagation of local excitations [8]. One way to overcome this is to engineer specific coupling Hamiltonians [7, 9–12]. However, the more a scheme relies on particular properties of the Hamiltonian, the more it will be affected by imperfections in its implementation [6]. A more general approach was taken in [13] where a specific encoding using time-dependent couplings at the sending and receiving ends of the chain achieved high fidelity transfer. Perfect transfer (i.e. unitary fidelity) for a whole class of unmodulated quantum chains was finally achieved in [14] by employing a parallel channel encoding where the sender of the message is able to transmit one qubit of information by operating on the first spins of two non-interacting copies of the chain. In quantum information theory, the ratio R between the number qubits that can be transferred with unitary fidelity and the number of channel copies used in the protocol is an important efficiency parameter [3, 15]. Therefore, one question that naturally arises is whether or not there is any special meaning in the $1/2$ value of R achieved in the protocol of [14]. More specifically, we pose the following question: can we use almost *any* quantum chain for perfect

and efficient (i.e. $R = 1$) quantum communication? In this paper, we give a sufficient and easily attainable condition for achieving this goal.

The paper is organized as follows: the model and the notation are introduced in section 2. The efficiency and the fidelity of the protocol are discussed in sections 3 and 4, respectively. Finally in section 5 we prove a theorem which provides us with a sufficient condition for achieving efficient and perfect state transfer in quantum chains.

2. The model

Consider a linear chain of N spins interacting through the Hamiltonian H. For $n = 1, \ldots, N$, we define the single excitation vector

$$|\boldsymbol{n}\rangle \equiv |00\cdots010\cdots0\rangle, \tag{1}$$

as the state of the chain in which the nth spin is in the computational base vector $|1\rangle$ and the remaining $N-1$ qubits are in the state $|0\rangle$. Analogously, we define $|\mathbf{0}\rangle \equiv |00\cdots0\rangle$ to be the state where all spins are in $|0\rangle$. We assume that $|\mathbf{0}\rangle$ is an eigenvector of H and that the N-dimensional subspace generated by the states $|\boldsymbol{n}\rangle$ is invariant under the time evolution $u(t) \equiv \mathrm{e}^{-\mathrm{i}Ht/\hbar}$, i.e.

$$|\boldsymbol{n}\rangle \longrightarrow u(t)|\boldsymbol{n}\rangle = \sum_{n'=1}^{N} f_{n',n}(t)|\boldsymbol{n}'\rangle, \tag{2}$$

where $f_{n',n}(t) \equiv \langle \boldsymbol{n}'|\mathrm{e}^{-\mathrm{i}Ht/\hbar}|\boldsymbol{n}\rangle$ is the probability amplitude that the excitation $|\boldsymbol{n}\rangle$ moves to $|\boldsymbol{n}'\rangle$ in the time interval t. A sufficient criterion for equation (2) is that H commutes with the z component of the total spin. A typical example is provided by a linear array of spins with Heisenberg interaction. In the original proposal of [1], one assumes that initially the chain is in $|\mathbf{0}\rangle$ and that at time $t = 0$ a first party (Alice) encodes one qubit of logical information in the first spin by preparing the chain in $|\Psi\rangle \equiv \alpha|\mathbf{0}\rangle + \beta|\mathbf{1}\rangle$ with α and β complex. By reading out the state of the Nth qubit at time t, a second party (Bob) will be able to recover the information transmitted.

Assume now that the two communicating parties operate on M independent (i.e. non-interacting) copies of the chain[3]. The idea is to use these copies to improve the overall fidelity of the communication. As in the original scheme [1], we assume Alice and Bob to control, respectively, the first and the last qubit of each chain (see figure 1). By preparing any superposition of her spins, Alice can in principle transfer up to M logical qubits. However, in order to improve the communication fidelity, the two parties will find it more convenient to redundantly encode only a small number (say $Q(M) \leqslant M$) of logical qubits in the M spins. By adopting these strategies, Alice and Bob are effectively sacrificing the efficiency $R(M) = Q(M)/M$ of their communication line in order to increase its fidelity. This is typical of any communication scheme and it is analogous to what happens in quantum error correction theory, where a single logical qubit is stored in many physical qubits. By focusing on those strategies that guarantee a (possibly asymptotic in M) unitary fidelity in the transmission of the $Q(M)$ encoded qubits, the efficiency $R(M)$ yields the capacity of the channel [15]. In the case of the quantum chains (2), it has been proved [14] the existence of an encoding of efficiency $R = 1/2$ which allows for unitary fidelity, by showing that for $M = 2$ it is possible to achieve perfect state transfer of a single logical qubit by using just two copies of the original

[3] This is quite a common attitude in quantum information theory [15] where successive uses of a memory-less channel are formally described by introducing many parallel copies of the channel (see [3] for a discussion on the possibility of applying this formal description to quantum chain models). Moreover, for the case at hand, the assumption of Alice and Bob dealing with 'real' parallel chain seems reasonable also from a practical point of view [16].

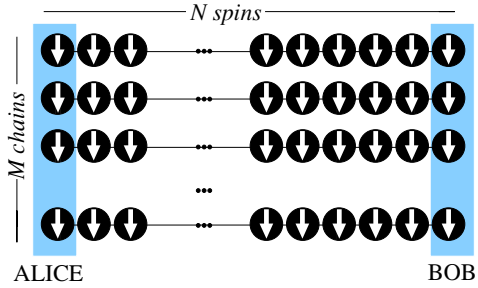

Figure 1. Schematic of the system: Alice and Bob operate M chains, each containing N spins. The spins belonging to the same chain interact through the Hamiltonian H which accounts for the transmission of the signal in the system. Spins of different chains do not interact. Alice encodes the information in the first spins of the chains by applying unitary transformations to her qubits. Bob recovers the message in the last spins of the chains by performing joint measurements.

chain. Here we will generalize such result by proving that given $M > 2$ there exist an optimal encoding–decoding strategy which asymptotically allows us to achieve perfect state transfer of $Q(M)$ qubits such that

$$\lim_{M\to\infty} R(M) = 1. \tag{3}$$

In other words, we show the possibility of achieving both perfect transfer *and* optimal efficiency.

Our strategy requires Alice to prepare superpositions of the M chains where $\sim M/2$ of them have a single excitation in the first location while the remaining are in $|\mathbf{0}\rangle$. Since in the limit $M \gg 1$ the number of qubit transmitted is $\log \binom{M}{M/2} \approx M$, this architecture guarantees optimal efficiency (3). On the other hand, our protocol requires Bob to perform collective measurements on his spins to determine if all the $\sim M/2$ excitations Alice is transmitting arrived at his location. We will prove that by repeating these detections many times, Bob is able to recover the messages with asymptotically perfect fidelity.

2.1. Notation

Before beginning the analysis, let us introduce some notation. In order to distinguish the M different chains we introduce the label $m = 1, \ldots, M$: in this formalism $|\boldsymbol{n}\rangle_m$ represents the state (1) of mth chain with a single excitation in the nth spin. In the following, we will be interested in those configurations of the whole system where K chains posses a single excitation while the remaining $M - K$ are in $|\mathbf{0}\rangle$, as in the case

$$|\mathbf{1}\rangle_1 \otimes |\mathbf{1}\rangle_2 \cdots \otimes |\mathbf{1}\rangle_K \otimes |\mathbf{0}\rangle_{K+1} \cdots \otimes |\mathbf{0}\rangle_M, \tag{4}$$

where, for instance, the first K chains have an excitation in the first chain location. Another more general example is given in figure 2. The complete characterization of these vectors is obtained by specifying (i) *which* chains possess a single excitation and (ii) *where* these excitations are located horizontally along the chains. In answering to the point (i) we introduce the K-element subsets S_ℓ, composed by the labels of those chains that contain an excitation. Each of these subsets S_ℓ corresponds to a subspace of the Hilbert space $\mathcal{H}(S_\ell)$ with a dimension N^K. The total number of such subsets is equal to the binomial coefficient $\binom{M}{K}$, which counts the number of possibilities in which K objects (excitations) can be distributed amongst M

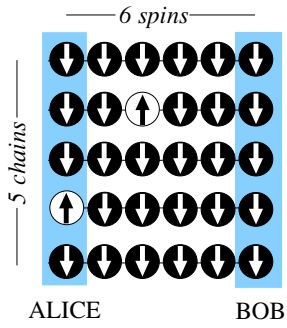

Figure 2. Example of our notation for $M = 5$ chains of length $N = 6$ with $K = 2$ excitations. The state above, given by $|\mathbf{o}\rangle_1 \otimes |\mathbf{3}\rangle_2 \otimes |\mathbf{o}\rangle_3 \otimes |\mathbf{1}\rangle_4 \otimes |\mathbf{o}\rangle_5$, has excitations in the chains $m_1 = 2$ and $m_2 = 4$ at the horizontal position $n_1 = 3$ and $n_2 = 1$. It is in the Hilbert space $\mathcal{H}(S_6)$ corresponding to the subset $S_6 = \{2, 4\}$ (assuming that the sets S_ℓ are ordered in a canonical way, i.e. $S_1 = \{1, 2\}$, $S_2 = \{1, 3\}$ and so on) and will be written as $|(3, 1); 6\rangle\rangle$. There are $\binom{5}{2} = 10$ different sets S_ℓ and the number of qubits one can transfer using these states is $\log_2 10 \approx 3$. The efficiency is thus given by $R \approx 3/5$, which is already bigger than in the original scheme [14].

parties (parallel chains). In particular, for any $\ell = 1, \ldots, \binom{M}{K}$, the ℓth subset S_ℓ will be specified by assigning its K elements, i.e. $S_\ell \equiv \left\{m_1^{(\ell)}, \ldots, m_K^{(\ell)}\right\}$ with $m_j^{(\ell)} \in \{1, \ldots, M\}$ and $m_j^{(\ell)} < m_{j+1}^{(\ell)}$ for all $j = 1, \ldots, K$. To characterize the location of the excitations, point (ii), we will introduce instead the K-dimensional vectors $\vec{n} \equiv (n_1, \ldots, n_K)$ where $n_j \in \{1, \ldots, N\}$. We can then define

$$|\vec{\boldsymbol{n}}; \ell\rangle\rangle \equiv \bigotimes_{j=1}^{K} |\boldsymbol{n_j}\rangle_{m_j^{(\ell)}} \bigotimes_{m' \in \overline{S}_\ell} |\mathbf{0}\rangle_{m'}, \tag{5}$$

where $\overline{S}_\ell$ is the complementary of S_ℓ to the whole set of chains. The state (5) represents a configuration where the jth chain of the subset S_ℓ is in $|\boldsymbol{n_j}\rangle$ while the chains that do not belong to S_ℓ are in $|\mathbf{0}\rangle$ (see figure 2 for an explicit example). The kets $|\vec{\boldsymbol{n}}; \ell\rangle\rangle$ are a natural generalization of the states $|\boldsymbol{n}\rangle_1 \otimes |\mathbf{0}\rangle_2$ and $|\mathbf{0}\rangle_1 \otimes |\boldsymbol{n}\rangle_2$ which were used for the 'dual-rail encoding' in [14]. They are useful for our purposes because they are mutually orthogonal, i.e.

$$\langle\langle \vec{\boldsymbol{n}}; \ell | \vec{\boldsymbol{n}}'; \ell' \rangle\rangle = \delta_{\ell\ell'} \delta_{\vec{n}\vec{n}'}, \tag{6}$$

and their time evolution under the Hamiltonian does not depend on ℓ (cf equation (12)). Among the vectors (5), those where all the K excitations are located at the beginning of the S_ℓ chains play an important role in our analysis. Here $\vec{n} = \vec{1} \equiv (1, \ldots, 1)$ and we can write

$$|\vec{\mathbf{1}}; \ell\rangle\rangle \equiv \bigotimes_{m \in S_\ell} |\mathbf{1}\rangle_m \bigotimes_{m' \in \overline{S}_\ell} |\mathbf{0}\rangle_{m'}. \tag{7}$$

According to equation (6), for $\ell = 1, \ldots, \binom{M}{K}$ these states form orthonormal set of $\binom{M}{K}$ elements. Analogously, by choosing $\vec{n} = \vec{N} \equiv (N, \ldots, N)$, we obtain the orthonormal set of $\binom{M}{K}$ vectors

$$|\vec{\boldsymbol{N}}; \ell\rangle\rangle \equiv \bigotimes_{m \in S_\ell} |\boldsymbol{N}\rangle_m \bigotimes_{m' \in \overline{S}_\ell} |\mathbf{0}\rangle_{m'}, \tag{8}$$

where all the K excitations are located at the end of the chains.

3. Efficient encoding

If all the M chains of the system are originally in $|\mathbf{0}\rangle$, the vectors (7) can be prepared by Alice by locally operating on her spins. Moreover, since these vectors span a $\binom{M}{K}$-dimensional subspace, Alice can encode in the chain $Q(M, K) = \log_2 \binom{M}{K}$ qubits of logical information by preparing the superpositions,

$$|\Phi\rangle\rangle = \sum_{\ell} A_\ell |\vec{\mathbf{1}}; \ell\rangle\rangle, \tag{9}$$

with A_ℓ complex coefficients. The efficiency of such encoding is hence $R(M, K) = \frac{\log_2 \binom{M}{K}}{M}$ which maximized with respect to K gives

$$R(M) = \frac{1}{M} \begin{cases} \log_2 \binom{M}{M/2}, & \text{for } M \text{ even}, \\ \log_2 \binom{M}{(M-1)/2}, & \text{for } M \text{ odd}. \end{cases} \tag{10}$$

The Stirling approximation can then be used to prove that this encoding is asymptotically efficient (3) in the limit of large M, e.g.

$$\log_2 \binom{M}{M/2} \approx \log_2 \frac{M^M}{(M/2)^M} = M. \tag{11}$$

Note that already for $M = 5$ the encoding is more efficient (cf figure 2) than in the 'dual-rail encoding' given in [14]. In the remainder of the paper, we show that the encoding (9) provides perfect state transfer by allowing Bob to perform joint measurements at his end of the chains.

4. Perfect transfer

Since the M chains do not interact with each other and possess the same free Hamiltonian H (this assumption can be relaxed, see [17]); the unitary evolution of the whole system is described by $U(t) \equiv \otimes_m u_m(t)$, with $u_m(t)$ being the operator (2) acting on the mth chain. The time evolved of the input $|\vec{\mathbf{1}}; \ell\rangle\rangle$ of equation (7) is thus equal to

$$U(t)|\vec{\mathbf{1}}; \ell\rangle\rangle = \sum_{\vec{n}} F[\vec{n}, \vec{1}; t]|\vec{n}; \ell\rangle\rangle, \tag{12}$$

where the sum is performed for all $n_j = 1, \ldots, N$ and

$$F[\vec{n}, \vec{n'}; t] \equiv f_{n_1, n'_1}(t) \cdots f_{n_K, n'_K}(t) \tag{13}$$

is a quantity which does *not* depend on ℓ. In equation (12) the term $\vec{n} = \vec{N}$ corresponds to having all the K excitations in the last locations of the chains. We can thus write

$$U(t)|\vec{\mathbf{1}}; \ell\rangle\rangle = \gamma_1(t)|\vec{N}; \ell\rangle\rangle + \sqrt{1 - |\gamma_1(t)|^2}|\boldsymbol{\xi}(t); \ell\rangle\rangle, \tag{14}$$

where

$$\gamma_1(t) \equiv \langle\langle \vec{N}; \ell | U(t) | \vec{\mathbf{1}}; \ell\rangle\rangle = F[\vec{N}, \vec{1}; t] \tag{15}$$

is the probability amplitude that all the K excitation of $|\vec{\mathbf{1}}; \ell\rangle\rangle$ arrive at the end of the chains, and

$$|\boldsymbol{\xi}(t); \ell\rangle\rangle \equiv \sum_{\vec{n} \neq \vec{N}} F_1[\vec{n}, \vec{1}; t]|\vec{n}; \ell\rangle\rangle, \tag{16}$$

with

$$F_1[\vec{n}, \vec{1}; t] \equiv \frac{F[\vec{n}, \vec{1}; t]}{\sqrt{1 - |\gamma_1(t)|^2}}, \tag{17}$$

is a superposition of terms where the number of excitations arrived to the end of the communication line is strictly less then K. It is worth noting that equation (6) yields the following relations:

$$\langle\!\langle \vec{N}; \ell | \boldsymbol{\xi}(t); \ell' \rangle\!\rangle = 0, \qquad \langle\!\langle \boldsymbol{\xi}(t); \ell | \boldsymbol{\xi}(t); \ell' \rangle\!\rangle = \delta_{\ell\ell'}, \tag{18}$$

which shows that $\{|\vec{|}\boldsymbol{\xi}(t); \ell\rangle\!\rangle\}$ is an orthonormal set of vectors which spans a subspace orthogonal to the states $|\vec{N}; \ell\rangle\!\rangle$. The time evolution of the input state (9) follows by linearity from equation (14), i.e.

$$|\Phi(t)\rangle\!\rangle = \gamma_1(t)|\Psi\rangle\!\rangle + \sqrt{1 - |\gamma_1(t)|^2}|\overline{\Psi}(t)\rangle\!\rangle, \tag{19}$$

with

$$|\overline{\Psi}(t)\rangle\!\rangle \equiv \sum_\ell A_\ell |\boldsymbol{\xi}(t); \ell\rangle\!\rangle, \qquad |\Psi\rangle\!\rangle \equiv \sum_\ell A_\ell |\vec{N}; \ell\rangle\!\rangle. \tag{20}$$

The vectors $|\Psi\rangle\!\rangle$ and $|\overline{\Psi}(t)\rangle\!\rangle$ are unitary transformations of the input message (9) where the orthonormal set $\{|\vec{\mathbf{1}}; \ell\rangle\!\rangle\}$ has been rotated into $\{|\vec{N}; \ell\rangle\!\rangle\}$ and $\{|\boldsymbol{\xi}(t); \ell\rangle\!\rangle\}$, respectively. Moreover, $|\Psi\rangle\!\rangle$ is the configuration we need to have for perfect state transfer at the end of the chain. In fact, it is obtained from the input message (9) by replacing the components $|\mathbf{1}\rangle$ (excitation in the first spin) with $|\boldsymbol{N}\rangle$ (excitation in the last spin). From equation (18) we know that $|\Psi\rangle\!\rangle$ and $|\overline{\Psi}(t)\rangle\!\rangle$ are orthogonal. This property helps Bob to recover the message $|\Psi\rangle\!\rangle$ from $|\Phi(t)\rangle\!\rangle$: he needs only to perform a collective measurement on the M spins he is controlling to establish if there are K or less excitations in those locations. The above is clearly a projective measure that can be performed without destroying the quantum coherence associated with the coefficients A_ℓ. Formally, this can described by introducing the observable

$$\Theta \equiv \mathbb{1} - \sum_\ell |\vec{N}; \ell\rangle\!\rangle\langle\!\langle \vec{N}; \ell|. \tag{21}$$

A single measure of Θ on $|\Phi(t_1)\rangle\!\rangle$ yields the outcome 0 with probability $p_1 \equiv |\gamma_1(t_1)|^2$, and the outcome +1 with probability $1 - p_1$. In the first case, the system will be projected in $|\Psi\rangle\!\rangle$ and Bob will get the message. In the second case, instead the state of the system will become $|\overline{\Psi}(t_1)\rangle\!\rangle$. Already at this stage the two communicating parties have a success probability equal to p_1. Moreover, as in [14], the channels have been transformed into a quantum erasure channel [18] where the receiver knows if the transfer was successful.

Consider now what happens if Bob fails to get the right answer from the measure. The state on which the chains is projected is explicitly given by

$$|\overline{\Psi}(t_1)\rangle\!\rangle = \sum_{\vec{n}\neq\vec{N}} F_1[\vec{n}, \vec{1}; t_1] \sum_\ell A_\ell |\vec{\boldsymbol{n}}; \ell\rangle\!\rangle. \tag{22}$$

Let us now consider the evolution of this state for another time interval t_2. By repeating the same analysis given above, we obtain an expression similar to (19), i.e.

$$|\Phi(t_2, t_1)\rangle\!\rangle = \gamma_2|\Psi\rangle\!\rangle + \sqrt{1 - |\gamma_2|^2}|\overline{\Psi}(t_2, t_1)\rangle\!\rangle, \tag{23}$$

where now the probability amplitude of getting all excitations in the Nth locations is described by

$$\gamma_2 \equiv \sum_{\vec{n}\neq\vec{N}} F[\vec{N}, \vec{n}; t_2] F_1[\vec{n}, \vec{1}; t_1].$$

In this case, $|\overline{\Psi}(t)\rangle\!\rangle$ is replaced by

$$|\overline{\Psi}(t_2, t_1)\rangle\!\rangle = \sum_\ell A_\ell |\boldsymbol{\xi}(t_2, t_1); \ell\rangle\!\rangle, \tag{24}$$

with

$$|\boldsymbol{\xi}(t_2,t_1);\ell\rangle\!\rangle = \sum_{\vec{n}\neq\vec{N}} F_2[\vec{n},\vec{1};t_2,t_1]|\vec{\boldsymbol{n}};\ell\rangle\!\rangle,$$

and F_2 defined as in equation (26) (see below). In other words, the state $|\Phi(t_2,t_1)\rangle\!\rangle$ can be obtained from equation (19) by replacing γ_1 and F_1 with γ_2 and F_2. Bob can hence try to use the same strategy he used at time t_1: i.e. he will check whether or not his M qubits contain K excitations. With (conditional) probability $p_2 \equiv |\gamma_2|^2$, he will get a positive answer and his quantum register will be projected in the state $|\Psi\rangle\!\rangle$ of equation (20). Otherwise, he will let the system evolve for another time interval t_3 and repeat the protocol. Reiterating the above analysis, it is possible to give a recursive expression for the conditional probability of success $p_q \equiv |\gamma_q|^2$ after $q-1$ successive unsuccessful steps. The quantity γ_q is the analogous of γ_2 and γ_1 of equations (15) and (23). It is given by

$$\gamma_q \equiv \sum_{\vec{n}\neq\vec{N}} F[\vec{N},\vec{n};t_q]F_{q-1}[\vec{n},\vec{1},t_{q-1},\ldots,t_1], \tag{25}$$

where

$$F_{q-1}[\vec{n},\vec{1};t_{q-1},\ldots,t_1] \equiv \sum_{\vec{n}'\neq\vec{N}} \frac{F[\vec{N},\vec{n}';t_{q-1}]}{\sqrt{1-|\gamma_{q-1}|^2}} F_{q-2}[\vec{n}',\vec{1};t_{q-2},\ldots,t_1] \tag{26}$$

and $F_1[\vec{n},\vec{1},t]$ is given by equation (17). In these equations $t_q,\ldots,t_1$ are the *time intervals* that occurred between the various protocol steps. Analogously, the conditional probability of failure at the step q is equal to $1-p_q$. The probability of having $j-1$ failures and a success at the jth step can thus be expressed as

$$\pi(j) = p_j(1-p_{j-1})(1-p_{j-2})\cdots(1-p_1), \tag{27}$$

while the total probability of success after q steps is obtained by the sum of $\pi(j)$ for all $j=1,\ldots,q$, i.e.

$$P_q = \sum_{j=1}^{q} \pi(j). \tag{28}$$

Since $p_j \geqslant 0$, equation (28) is a monotonic function of q. As a matter of fact, in the next section we prove that under a very general hypothesis on the system Hamiltonian, the probability of success P_q converges to 1 in the limit of $q\to\infty$. This means that by repeating many times the collective measure described by Θ, Bob is guaranteed to get, sooner or later, the answer 0 and hence the message Alice sent to him. In other words, our protocol allows perfect state transfer in the limit of repetitive collective measures. Note that the above analysis applies for all classes of subsets S_ℓ. The only difference between different choices of K is in the velocity of the convergence of $P_q \to 1$. In any case, by choosing $K \sim M/2$, Alice and Bob can achieve perfect fidelity *and* optimal efficiency.

5. Convergence theorem

Here we show that if there exists no eigenvector $|e_m\rangle$ of the quantum chain Hamiltonian H which is orthogonal to $|\boldsymbol{N}\rangle$, then there is a choice of the time intervals $t_q, t_{q-1},\ldots,t_1$ such that P_q of equation (28) converges to 1 in the limit of $q\to\infty$. For the special case $M=2$ and $K=1$, this was numerically shown in [14].

The state of the system at a time interval of t_q after the $(q-1)$th failure can be expressed in compact form as follows:

$$|\Phi(t_q,\ldots,t_1)\rangle\!\rangle = \frac{U(t_q)\Theta U(t_{q-1})\Theta\cdots U(t_1)\Theta|\Phi\rangle\!\rangle}{\sqrt{(1-p_{q-1})\cdots(1-p_1)}},$$

with $U(t)$ the unitary time evolution generated by the system Hamiltonian and Θ the projection defined in equation (21). One can verify, for instance, that for $q=2$, the above equation coincides with equation (23). (For $q=1$, this is just (19) evaluated at time t_1.) By definition, the conditional probability of success at the qth step is equal to

$$p_q \equiv |\langle\!\langle\Psi|\Phi(t_q,\ldots,t_1)\rangle\!\rangle|^2.$$

Therefore, equation (27) yields

$$\begin{aligned}\pi(q) &= |\langle\!\langle\Psi|U(t_q)\Theta U(t_{q-1})\Theta\cdots U(t_1)\Theta|\Phi\rangle\!\rangle|^2\\ &= |\langle\!\langle\vec{N};\ell|U(t_q)\Theta U(t_{q-1})\Theta\cdots U(t_1)\Theta|\vec{\mathbf{1}};\ell\rangle\!\rangle|^2,\end{aligned} \tag{29}$$

where the second identity stems from the fact that, according to equations (2) and (6), $U(t)\Theta$ preserves the orthogonality relation among states $|\vec{n};\ell\rangle\!\rangle$ with distinct values of ℓ. Analogously to the cases of equations (13) and (15), the second identity of (29) establishes that $\pi(q)$ can be computed by considering the transfer of the input $|\vec{\mathbf{1}};\ell\rangle\!\rangle$ for *arbitrary* ℓ. The expression (29) can be further simplified by noting that for a given ℓ the chains of the subset $\overline{S}_\ell$ contribute with a unitary factor to $\pi(q)$ and can be thus neglected (according to (7) they are prepared in $|\mathbf{0}\rangle$ and do not evolve under $U(t)\Theta$). Identify $|\vec{\mathbf{1}}\rangle\!\rangle_\ell$ and $|\vec{N}\rangle\!\rangle_\ell$ with the components of $|\vec{\mathbf{1}};\ell\rangle\!\rangle$ and $|\vec{N};\ell\rangle\!\rangle$ relative to the chains belonging to the subset S_ℓ. In this notation, we can rewrite equation (29) as

$$\pi(q) = |_\ell\langle\!\langle\vec{N}|U_\ell(t_q)\Theta_\ell\cdots U_\ell(t_1)\Theta_\ell|\vec{\mathbf{1}}\rangle\!\rangle_\ell|^2, \tag{30}$$

where $\Theta_\ell = \mathbb{1}_\ell - |\vec{N}\rangle\!\rangle_\ell\langle\!\langle\vec{N}|$ and $U_\ell(t)$ is the unitary operator $\otimes_{m\in S_\ell}u_m(t)$ which describes the time evolution of the chains of S_ℓ. Furthermore, since all the K chains of S_ℓ contain exactly one excitation and U_ℓ preserves the total number of excitations, we only need to consider the N^{2K}-dimensional restrictions of this operator in the corresponding subspace $\mathcal{H}(\mathcal{S}_\ell)$.

To prove that there exists a suitable choice of t_j such that the series (28) converges to 1, it is sufficient to consider the case $t_j=\tau>0$ for all $j=1,\ldots,q$: this is equivalent to selecting decoding protocols with constant measuring intervals.

By introducing the operator $T_\ell \equiv U_\ell(\tau)\Theta_\ell$, equation (30) thus becomes

$$\begin{aligned}\pi(q) &= |_\ell\langle\!\langle\vec{N}|(T_\ell)^q|\vec{\mathbf{1}}\rangle\!\rangle_\ell|^2\\ &= {}_\ell\langle\!\langle\vec{\mathbf{1}}|\left(T_\ell^\dagger\right)^q|\vec{N}\rangle\!\rangle_\ell\langle\!\langle\vec{N}|(T_\ell)^q|\vec{\mathbf{1}}\rangle\!\rangle_\ell = w(q)-w(q+1),\end{aligned} \tag{31}$$

where

$$w(j) \equiv {}_\ell\langle\!\langle\vec{\mathbf{1}}|\left(T_\ell^\dagger\right)^j(T_\ell)^j|\vec{\mathbf{1}}\rangle\!\rangle_\ell = \|(T_\ell)^j|\vec{\mathbf{1}}\rangle\!\rangle_\ell\|^2 \tag{32}$$

is the norm of the vector $(T_\ell)^j|\vec{\mathbf{1}}\rangle\!\rangle_\ell$. Substituting equation (31) in equation (28) yields

$$P_q = \sum_{j=1}^{q}[w(j)-w(j+1)] = 1-w(q+1), \tag{33}$$

where the property $w(1) = {}_\ell\langle\!\langle\vec{\mathbf{1}}|\Theta_\ell|\vec{\mathbf{1}}\rangle\!\rangle_\ell = 1$ was employed. Proving the thesis is hence equivalent to prove that for $q\to\infty$ the succession $w(q)$ nullifies. This last relation can be studied using properties of power-bounded matrices [19]. In fact, by introducing the norm of the operator $(T_\ell)^q$, we have

$$w(q) = \|(T_\ell)^q|\vec{\mathbf{1}}\rangle\!\rangle_\ell\|^2 \leqslant \|(T_\ell)^q\|^2 \leqslant c\left(\frac{1+\rho(T_\ell)}{2}\right)^{2q}, \tag{34}$$

where c is a positive constant which does not depend[4] on q and where $\rho(T_\ell)$ is the spectral radius of T_ℓ, i.e. the eigenvalue of T_ℓ with maximum absolute value (NB: even when T_ℓ is not diagonalizable this is a well-defined quantity). Equation (34) shows that $\rho(T_\ell) < 1$ is a sufficient condition for $w(q) \to 0$. In our case we note that, given any normalized eigenvector $|\lambda\rangle\!\rangle_\ell$ of T_ℓ with eigenvalue λ, we have

$$|\lambda| = \|T_\ell|\lambda\rangle\!\rangle_\ell\| = \|\Theta_\ell|\lambda\rangle\!\rangle_\ell\| \leqslant 1, \tag{35}$$

where the inequality follows from the fact that Θ_ℓ is a projector. Note that in equation (35) the identity holds only if $|\lambda\rangle\!\rangle_\ell$ is also an eigenvector of Θ_ℓ with eigenvalue +1, i.e. only if $|\lambda\rangle\!\rangle_\ell$ is orthogonal to $|\vec{\boldsymbol{N}}\rangle\!\rangle_\ell$. By definition, $|\lambda\rangle\!\rangle_\ell$ is an eigenvector of $T_\ell = U_\ell(\tau)\Theta_\ell$: therefore, the only possibility to have the equality in equation (35) is that (i) $|\lambda\rangle\!\rangle_\ell$ is an eigenvector of $U_\ell(\tau)$ (i.e. an eigenvector of the Hamiltonian H_ℓ^{tot} of the chain subset S_ℓ) and (ii) it is orthogonal to $|\vec{\boldsymbol{N}}\rangle\!\rangle_\ell$. By negating the above statement, we get a sufficient condition for the thesis. Namely, if all the eigenvectors $|\vec{E}\rangle\!\rangle_\ell$ of H_ℓ^{tot} are not orthogonal to $|\vec{\boldsymbol{N}}\rangle\!\rangle_\ell$, then the absolute values of the eigenvalues λ of T_ℓ are strictly smaller than 1 which implies $\rho(T_\ell) < 1$ and hence the thesis. Since the S_ℓ channels are identical and do not interact, the eigenvectors $|\vec{E}\rangle\!\rangle_\ell \equiv \bigotimes_{m\in S_\ell} |e_m\rangle_m$ are tensor product of eigenvectors $|e_m\rangle$ of the single-chain Hamiltonians H. Using the notation introduced in equation (1), the sufficient condition becomes

$${}_\ell\langle\!\langle \vec{E}|\vec{\boldsymbol{N}}\rangle\!\rangle_\ell = \prod_{m\in S_\ell} {}_m\langle \boldsymbol{N}|e_m\rangle_m \neq 0, \tag{36}$$

which can be satisfied only if $\langle \boldsymbol{N}|e_m\rangle \neq 0$ for all eigenvectors $|e_m\rangle$ of the single-chain Hamiltonian H in the single excitation sector.

While we have proved here that for equal time intervals the probability of success is converging to unity, in practice one may use *optimal* measuring time intervals for a faster transfer [14]. We also point out that timing errors may delay the transfer, but will not decrease the asymptotic fidelity.

5.1. Quantum chains with nearest-neighbour interactions

It is worth noting that equation (36) is a very weak condition, which is satisfied for *any* open nearest-neighbour quantum chain as long as the transition amplitude $f_{1,N}(t)$ from Alice to Bob (cf equation (2)) is not identical to zero. Let us prove this by contradiction: assume there exists a normalized eigenvector $|e_m\rangle$ of the single-chain Hamiltonian H such that

$$\langle \boldsymbol{N}|e_m\rangle = 0. \tag{37}$$

Because $|e_m\rangle$ is an eigenstate, we can conclude that also

$$\langle e_m|H|\boldsymbol{N}\rangle = 0. \tag{38}$$

If we act with the Hamiltonian on the ket in equation (38), we may get some term proportional to $\langle e_m|\boldsymbol{N}\rangle$ (corresponding to an Ising-like interaction) and some part proportional to $\langle e_m|\boldsymbol{N}-1\rangle$ (corresponding to a hopping term; if this term did not exist, then clearly $f_{1,N}(t) = 0$ for all times). We can thus conclude that

$$\langle e_m|\boldsymbol{N}-1\rangle = 0. \tag{39}$$

Note that for a closed chain, e.g. a ring, this need not be the case, because then also a term proportional to $\langle e_m|\boldsymbol{N}+1\rangle = \langle e_m|\boldsymbol{1}\rangle$ would occur. If we insert the Hamiltonian into

[4] If S is the similarity transformation that puts T_ℓ into the Jordan canonical form, i.e. $J = S^{-1}T_\ell S$, then c is given explicitly by $c = \|S\|\,\|S^{-1}\|$.

equation (39) again, we can use the same reasoning to see that

$$\langle e_m|\boldsymbol{N}-2\rangle = \cdots = \langle e_m|\mathbf{1}\rangle = 0 \tag{40}$$

and hence $|e_m\rangle = 0$, which is a contradiction to $|e_m\rangle$ being normalized. We thus conclude that any nearest-neighbour Hamiltonian that can transfer quantum information with nonzero fidelity (including the Heisenberg chains analysed in [1, 3]) is capable of efficient and perfect transfer when used in the context of parallel chains.

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